Lecture 5

This document provides practice problems that are similar to those that will be asked during the final exam. Please note that the document reflects the style and not the number of the questions that will be on the exam.

Problem 1

Consider the scalar continuous-time dynamical system described by $\dot{x} = \cos(x)$. What can we say about its equilibria?

(a) The points $\{x_k = \pi/2 + \pi k, k \in \mathbb{Z}\}$ are unstable equilibria.

(b) The points $\{x_k = \pi/2 + \pi k, k \in \mathbb{Z}\}$ are stable equilibria.

(c) The points $\{x_k = \pi/2 + 2\pi k, k \in \mathbb{Z}\}$ are unstable equilibria. The points $\{x_k = (3/2)\pi + 2\pi k, k \in \mathbb{Z}\}$ are stable equilibria.

(d) The points $\{x_k = \pi/2 + 2\pi k, k \in \mathbb{Z}\}$ are stable equilibria. The points $\{x_k = (3/2)\pi + 2\pi k, k \in \mathbb{Z}\}$ are unstable equilibria.

(e) The points $\{x_k = \pi/2 + \pi k, k \in \mathbb{Z}\}$ are not equilibrium points.

Problem 2

Consider the scalar continuous-time dynamical system described by $\dot{x} = \cos(x)$. What can we say about a trajectory that starts at the origin at time $t = 0$?

(a) The trajectory is not unique.

(b) The trajectory oscillates between $-\pi/2$ and $\pi/2$.

(c) The trajectory approaches $\pi/2$ for $t \longrightarrow \infty$.

(d) The trajectory approaches $-\pi/2$ for $t \longrightarrow \infty$.

Problem 3 (answer corrected on December 8, 2021)

Are there limit points of continuous-time dynamical systems that are not equilibria?

(a) No.

(b) Yes.

Problem 4

Consider the scalar continuous-time dynamical system $\dot{x} = -0.5x - \sin(x)$. Which of the following statements are correct?

(a) $V(x) = 0.25x^2 - \cos(x) + 1$ is a Lyapunov function for this system.

(b) $V(x) = -0.25x^2 + \cos(x) - 1$ is a corresponding Lyapunov function.

(c) The equilibrium at the origin is unstable.

Problem 5

We analyze gradient-flow dynamics on the objective function $-\cos(x)$, which leads to the dynamical system $\dot{x} = -\sin(x)$. Does the following claim on the convergence rate hold?

$$|x(t)| \leq 10 \cdot |x(0)| \cdot \exp(-t) \quad \forall t \geq 0 \text{ and } \forall x(0) \in (-\pi, \pi).$$

(a) Yes.

(b) No.

Problem 6

We recall that the space L_{2e} is defined as

$$L_{2e} := \left\{ g : [0, \infty) \to \mathbb{R} \;\middle|\; \int_0^T g(t)^2 \, \mathrm{d}t < \infty, \; \forall T \geq 0 \right\}.$$

Which of the following statements are true?

(a) The function $f(t) = e^t$ is an element of L_{2e}.

(b) The function $f(t) = t^3$ is an element of L_{2e}.

(c) The function $f(t) = 1/(t-1)$ is an element of L_{2e}.

(d) The function $f(t) = 1/|t-1|^{1/4}$ is an element of L_{2e}.

Note: Assume that all functions f here are restricted to the domain $[0, \infty)$.